Cohesive solutions of intersonic moving dislocations

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Abstract
A class of cohesive solutions of moving glide dislocations with intersonic speeds has been derived on the basis of the fundamental equation of a moving dislocation introduced by Weertman in conjunction with a proposed generalized Bilby–Cottrell–Swinden–Dugdale model. In this model we assume a straight weak path within an infinite elastic plate. Two length scales, namely the width (thickness) of the weak path and the material intrinsic length, which scales strain-gradient-induced hardening and energy dissipation, are taken into account by applying the traction–separation law for the decohesion of the weak path. Dislocations propagate along this weak path with a speed higher than the shear wave speed. The accumulation of these moving dislocations forms a macroscale crack growth with a cohesive zone ahead of the crack tip. Similar to the Bilby–Cottrell–Swinden–Dugdale model, the remote enforced stress and/or stress-rate boundary conditions are represented as an equivalent crack surface traction associated with the dislocation distribution. The involved Cauchy integral and corresponding eigenvalue problem are solved using the algorithms introduced by Muskhelishvili and by Weertman. The problems associated with three types of decohesion law are constant traction, traction linearly dependent on separation, and separation- and separation-rate-dependent traction. These problems are solved using three different solution strategies: the direct-integration method, the iteration method and the Jacobi polynomial expansion respectively. The derived solutions provide explicit relations between the remote load propagation speed, the material intrinsic length, the weak path thickness and the strain-rate-hardening parameter. The solutions demonstrate that the intersonic speed region can be divided into two subdomains; steady-state propagation occurs within the subdomain where the propagation speeds are equal to or greater than the Eshelby speed (\(c_s \times 2^{1/2}\), where \(c_s\) is the shear wave speed). For a weak path with a finite width and the corresponding decohesion law scaled by material intrinsic length, an intersonic crack propagation will not take place if only a constant remote stress is imposed. A 'steady-state' crack surface load and/or remote stress-rate boundary condition, which can be considered as a point force or a distributed force with a constant distance to the moving crack tip, is required to maintain steady-state intersonic crack propagation.

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§ 1. Introduction

Starting in the 1950s, intersonic dislocation motion was analysed by Frank (1949), Eshelby (1949), Weertman (1969), Weertman and Weertman (1980) and recently by Gao et al. (1999a). Here we use the term ‘intersonic’ to refer to crack growth speeds between the shear wave speed $c_s$ and the dilatation wave speed $c_d$ whereas the term ‘subsonic’ refers to crack speeds below the Rayleigh wave speed $c_R$ and the term ‘supersonic’ refers to speeds greater than $c_d$. When a dislocation propagates with a constant speed, the fundamental governing equations of glide, climb and screw dislocation motions at subsonic, intersonic and supersonic speeds have been derived by Weertman (1969) and Weertman and Weertman (1980). For a glide dislocation, the Weertman equation indicates that at the intersonic region the Eshelby speed, that is $2^{1/2}c_s$, is the favourite speed for its motion. This coincides with the linear elastic asymptotic analysis of crack propagation (Freund 1979). Considering the slipping-friction process ahead of an intersonic shear fracture, Burridge et al. (1979) first proposed a cohesive-like friction slipping model. By presuming a cohesive zone size and constant stress distribution, Broberg (1999b) derived the intersonic crack solutions in orthotropic material. Under static loading, Rice and Thomson (1974) proposed that the initiation of a brittle crack growth can be modelled as the spontaneous emission of dislocations from the crack tip.

Recently, seismic analysis (Bouchon et al. 2001) and laboratory-scale experiments (Rosakis et al. 1999) demonstrated that, in layered materials, shear failure can propagate more rapidly than a shear wave when it grows along a preferred weak path. Indeed, the limitation of the propagation speed and the theoretical prediction of such a crack growth are topics of interest in the field of fracture mechanics (Andrews 1976, Burridge et al. 1979, Freund 1979, Broberg 1999a, Gumbsch and Gao 1999, Needleman 1999, Needleman and Rosakis 1999, Ben-Zion 2001, Bouchon et al. 2001, Gao et al. 2001, Huang and Gao 2001, Rosakis 2001, Kubair and Geubelle 2002, 2003, Samudrala et al. 2002). A detailed review of the recent progress of subsonic and intersonic failure in seismological analysis has been given by Rosakis (2001). A comprehensive numerical study of intersonic cracks has been performed by Hao et al. (2003b).

For fast crack propagation along a preferred weak path, the decohesion and debonding within this path dominate the propagation. The following questions are of interest.

(i) How can the relationship between weak path mechanisms and the surrounding stress–strain fields be established?

(ii) As an intersonic crack growth occurs only within a weak preferred path, what are the effects of length scales, such as the width of this path or material intrinsic length, on such crack growth?

(iii) How can the propagation speed in the intersonic region be determined?

(iv) What is the effect of a stress-rate boundary condition on intersonic crack growth?

(v) What are the effects of strain gradient and material strain-rate sensitivity?

Focusing on these problems, in the present work a dislocation-mechanism-based crack propagation model, called the generalized Bilby–Cottrell–Swinden–Dugdale (GBCSD) model, is proposed by combining the fundamental equation of moving dislocation introduced by Weertman (1969) and Weertman and Weertman (1980)
and the Bilby–Cottrell–Swinden (1963)–Dugdale (1960) (BCSD) model. The corresponding solutions are obtained. The contributions that we try to present in this work are as follows.

(i) A GBCSD model, which establishes the relationships between (any type of) remote applied stress and applied stress rate, crack surface tractions, fracture energy dissipation and crack propagation speed, is developed.

(ii) A traction–separation law based on strain gradient theory is applied in the cohesive solution that includes both material extrinsic and intrinsic lengths, where the extrinsic length is the thickness of the weak path and the intrinsic length scales the Taylor hardening and energy dissipation during material failure.

(iii) A solution procedure using Jacobi polynomials is presented which can be applied to the class of eigenvalue problems with a Cauchy-type singular integration.

(iv) The analytical solution provides the relationship between the propagation speed, material intrinsic length, weak path thickness, strain-rate-hardening parameters and remote loads.

This paper is organized as follows: The next section introduces the proposed GBCSD model, from which a boundary value problem is established. In §3 the solution procedure and governing equation are introduced. The intersonic crack growth solution with constant cohesive traction is presented in §4. For the problem with strain-gradient-based traction–separation law, the solution procedure, results and discussion are given in §5. The conclusions are presented in §6.


2.1. Assumptions and Governing Equations

A static crack in an elastic solid can be treated as the accumulation of dislocations (Dugdale 1960, Bilby et al. 1963). Rice and Thomson (1974) modelled a brittle crack growth initiation as a dislocation emission (figure 1 (a)). In the present work, the following hypotheses are employed.

**Hypothesis 1.** A moving glide dislocation can be smeared out into a finite domain, expressed as a dislocation density within this domain.

**Hypothesis 2.** A dynamic crack growth is considered as the accumulation of such smeared-out moving dislocations.

**Hypothesis 3.** A crack propagates along a straight weak path within an elastic plane. A two-dimensional plane stress–strain condition is assumed.

**Hypothesis 4.** This weak path has a height \( h \), defined as the domain \( -\infty < X_1 < \infty; |X_2| < h/2 \), where \( \{X_I\} \) is a moving coordinate system originating at the crack tip.

**Hypothesis 5.** The dislocation density function is constant along the \( X_2 \) direction within the weak path; when \( h \) takes the value of the Burgers vector, the weak path allows only one glide dislocation in the \( X_2 \) direction.

**Hypothesis 6.** There is steady-state propagation.
Hypothesis 7. This is the localized principle (Saint-Venant’s principle): as the elastic solid contains a weak path of intersonic crack–dislocation propagation, the effect of crack growth on the stress–strain distribution is significant only within the domain that contains the crack tip and is characterized by a finite length $L_R$. Beyond this domain, the stress–strain field can be considered as unchanged during this crack propagation, assuming that the size of this elastic solid is much larger than $L_R$.

These assumptions can be interpreted as a dislocation-accumulation-induced intersonic crack growth model, illustrated in figure 1 (b) where $\{X_1\}$ is the moving coordinate system originating at the crack tip. The crack lies on the negative $X_1$ axis with two traction-free surfaces and the interval $0 \leq X_1 \leq \Delta a_c$ is defined as the

Figure 1. (a) The Rice–Thomson model with the Weertman smeared-out intersonic dislocation along a weak path emission. (b) Proposed model: the accumulation of the moving dislocations (shown in (a)) forms an intersonic cohesive zone and crack growth along a weak path with a finite width and obeying cohesive law based on strain gradient theory.
cohesive zone where the opening is governed by a traction separation law. For this problem, the Weertman moving dislocation equation is written in the form

\[
\sigma_{12}(X_1) = \alpha_1 D(X_1) + \alpha_2 \int_{-\infty}^{\Delta \xi_c} \frac{D(w)}{X_1 - w} \, dw, \tag{1}
\]

where \(\sigma_{12}\) denotes the shear stress along the \(X_1\) axis, and \(\alpha_1\) and \(\alpha_2\) are constants given by

\[
\alpha_1 = -\frac{2\mu c_\xi^2 \alpha^4}{V^2 v_1^s}, \quad \alpha_2 = \frac{2\mu c_\xi^2 v_2}{V^2 \pi}, \tag{2}
\]

with

\[
\alpha = \left(1 - \frac{V^2}{2c_\xi^2}\right)^{1/2}, \quad v_1^s = \left(\frac{V^2}{c_\xi^2} - 1\right)^{1/2}, \quad v_2 = \left(1 - \frac{V^2}{c_d^2}\right)^{1/2},
\]

where \(\mu\) is the Lamé constant. \(D(X_1)\) in equation (1) is the unknown dislocation density that is associated with the displacement jump over the weak path, denoted by \([u_1]\):

\[
D(X_1) = -2[u_1(X_1)]_1. \tag{3a}
\]

When a glide dislocation moves along the weak path, the thickness \(h\) of the weak path takes the value of the Burgers vector. Hence

\[
D(X_1) = -2[u_1(X_1)]_1 = -2 \frac{d}{dX_1} \left[ u_1 \left( X_1, \frac{h}{2} \right) - u_1 \left( X_1, -\frac{h}{2} \right) \right]. \tag{3b}
\]

As illustrated in figure 1(b), the bonding layer with a thickness \(h\) forms a ‘preferred weaker path’ within \(X_2 \leq \pm h/2\), which guides the crack propagation with a speed \(V\). The rest of the parts of the body are linear elastic. When the value is much smaller than other scales, conventionally the weak path can be considered as a mathematical ‘line’ for the global field solution.

The moving coordinate system \(\{X_I\}\), which originates at the crack, is related to the spatial coordinate system \(\{x_i\}\) through the Galilean transformation

\[
X_1 = x_1 - Vt, \quad X_2 = x_2. \tag{4}
\]

The time derivative by holding \(\{x_1, x_2\}\) fixed can be expressed in the coordinate system governed by equation (4). For any function \(f(t, x_i) = f(X_1)\), using the chain rule

\[
\frac{\partial f(t, x_i)}{\partial t} = -V \frac{\partial f(X_1)}{\partial X_1}. \tag{5}
\]

Hence, the displacement, stress and strain fields defined in a moving coordinate system \(\{X_I\}\) can be expressed in terms of \(X_1\) only.

In equation (1), \(\alpha_1 D(X_1)\) is directly associated with the change in \(\sigma_{12}\), which radiates shock waves when the dislocation moves with intersonic speeds. Hence, this term characterizes the ‘supersonic’ motion of the dislocation with respect to the shear wave speed, whereas the remaining term characterizes the subsonic motion with respect to the dilatation wave speed. It is noteworthy that \(\alpha_1\) vanishes at the Eshelby speed, that is \(V = 2^{1/2} c_\xi\). This coincides with the radiation-free velocity obtained in linear fracture mechanics analysis (Freund 1979).
2.2. The cohesive law

A linearized expression of the traction–separation law based on strain gradient theory (Hao et al. 2000a,b) is applied to the cohesive zone in figure 1 (b):

\[
\sigma_{12}^{\text{coh}}(X_1) = \tau_{\text{th}} \left[ 1 - \pi_1 \left[ u_1(X_1) \right] + \left( \pi_1^a \frac{l}{h} + \pi_1^b \frac{mV}{h} \right) D(X_1) \right] \quad \text{for } 0 \leq X_1 \leq \Delta a_c, \quad (6)
\]

where \( \tau_{\text{th}} \) is the static material shear strength of the bonding layer. The material intrinsic length \( l \), defined as (Gao et al. 1999b)

\[
l = 3^{1/2} \frac{Eb}{\tau_{\text{th}}},
\]

scales the second term on the right-hand side of equation (6), that is the weakening of traction due to the separation \([u_1(X_1)]\). The third and fourth terms on the right-hand side of equation (6) represent the effects of strain gradient and strain-rate hardening respectively, where \( m \) is the material viscosity exponent which ranges from 0 to 1; \( \pi_1, \pi_1^a \) and \( \pi_1^b \) are dimensionless constants (see appendix A).

The original form of the traction–separation law introduced by Hao et al. (2000a,b) is

\[
\sigma_{12}^{\text{coh}}(X_1) = \tau_{\text{th}} \tilde{T}([u_1], l) \left[ f^2(\tilde{\epsilon}) + lt \right]^{1/2} \left( 1 + \frac{\dot{\tilde{\epsilon}}}{\tilde{\epsilon}_0} \right)^m
\]

where \( \tilde{T}([u_1], l) \) is the normalized traction from a theoretical solution. An empirical formula for \( \tilde{T}([u_1], l) \) based on curve fitting is

\[
\tilde{T}([u_1], l) = \frac{-0.5398 Y^2 + 1.5867 Y - 0.0466}{1 - k_{tr}},
\]

with

\[
Y = \exp \left[ 10^6 \left( \frac{u_l}{l} \right)^{2.7} \right].
\]

The other two terms in equation (8) are the Taylor strain-gradient-hardening law (Gao et al. 1999b) and the strain-rate-hardening law (Xia and Shih 1996), respectively. Equation (6) is the Taylor expansion of equation (8) by leaving out the terms of \( O(l^k) \) where \( k \geq 2 \).

In the weak path shown in figure 1 (b), we have

\[
\varepsilon_{12} = \frac{[u_1]}{h} \quad \text{and} \quad \varepsilon_{ij} = O(\varepsilon_{12}) \quad \text{for } i = j. \quad (9a)
\]

Hence, the equivalent strain rate \( \dot{\tilde{\epsilon}} \) and the equivalent strain gradient \( \hat{\eta} \) in equation (8) can be expressed as

\[
\dot{\tilde{\epsilon}} = -V \frac{\partial}{\partial X_1} \left( \frac{2}{3} \varepsilon_{ij} \varepsilon_{ij} \right)^{1/2}
\]

\[
\approx -V \frac{\partial}{\partial X_1} \left( \frac{2}{3} \varepsilon_{12} \varepsilon_{12} \right)^{1/2}
\]

\[
= -V [u_1]_1 \left( \frac{2}{3} \right)^{1/2}
\]

\[
= D(X_1) \frac{V}{h} \left( \frac{1}{6} \right)^{1/2}
\]

\[
\hat{\eta} = \frac{\partial}{\partial X_1} \left( \varepsilon_{ij} \varepsilon_{ij} \right)^{1/2}
\]

\[
\approx \frac{\partial}{\partial X_1} \left( \varepsilon_{12} \varepsilon_{12} \right)^{1/2}
\]

\[
= D(X_1) \frac{V}{h} \left( \frac{1}{6} \right)^{1/2}
\]
and

\[
\eta = \left( \frac{1}{4} u_{i,jk} u_{i,jk} \right)^{1/2} = \left( \frac{1}{2} \varepsilon_{12,1} \varepsilon_{12,1} + O(\varepsilon_{12,1} \varepsilon_{12,1}) \right)^{1/2} \approx \left[ \frac{1}{2} \left( \frac{|u_{i,j}|}{h} \right)^2 \right]^{1/2} = \frac{D(X_1)}{h} \left( \frac{1}{8} \right)^{1/2},
\]

respectively. The constants \( \sigma^a_2 \) and \( \sigma^b_2 \) in (6) can be determined through equations (8) and (9). The detailed expressions for these are given in appendix A.

When the decohesion process does not depend upon strain rate and strain gradient, equation (6) becomes

\[
\sigma^{\text{coh}}_{12}(X_1) = \tau_{\text{th}} \left( 1 - \frac{\pi_1}{l} |u_1(X_1)| \right) \text{ for } 0 \leq X_1 \leq \Delta a_c. \quad (10a)
\]

A degenerated form of equation (10a) is

\[
\sigma^{\text{coh}}_{12}(X_1) = \tau_{\text{th}}. \quad (10b)
\]

2.3. Remote boundary condition: Bilby–Cottrell–Swinden–Dugdale superposition

The remote boundary condition is taken into account by applying the BCSD superposition (Dugdale 1960, Bilby et al. 1963), as shown in figure 2. The shear stress introduced in the moving dislocation equation, that is the left-hand side of equation (1), yields

\[
\sigma_{12}(X_1) = \sigma^{\text{coh}}_{12}(X_1) - \tau^{\text{remote}}_{\text{eq}}(X_1), \quad (11)
\]

where \( \tau^{\text{remote}}_{\text{eq}}(X_1) \) denotes the stress imposed on the traction-free crack surface and cohesive zone owing to the remote boundary condition. The distribution of \( \sigma_{12}(X_1) \) is depicted in the inset in figure 2(a).

In the BCSD model, \( \tau^{\text{remote}}_{\text{eq}}(X_1) \) has the same value as the remote applied stress that must be less than or equal to \( \tau_{\text{th}} \), the static flow strength of the material. For dynamic fracture, the remote boundary condition can be either an enforced stress or an enforced stress rate; it could be either a distributed stress or a concentrated force. They may have equal effects on the crack propagation. That is why we propose an equivalent remote stress \( \tau^{\text{remote}}_{\text{eq}}(X_1) \) that is expressed as

\[
\tau^{\text{remote}}_{\text{eq}}(X_1) = \tau_{\text{remote}} + \tau_{\text{d remote}}^{\text{remote}} D(X_1) \quad \text{for } X_1 \leq \Delta a_c, \quad (12)
\]

where \( \tau_{\text{remote}} \) is a constant. It can be, for example,

\[
\tau_{\text{remote}} = \tau_{\infty}, \quad (13)
\]

that is the remote imposed uniform shear stress.
The term \( \tau^\text{remote}_d D(X_1) \) reflects the effects of a non-uniformly distributed remote stress, a stress-rate boundary condition, a concentrated load and the stress imposed on the crack surface. The balance of energy dissipation requires that

\[
\frac{(K^\text{dyn}_{II})^2}{E} = \frac{1}{2} \int_{-L}^{\Delta a_c} \tau^\text{eq}(X_1) D(X_1) \, dX_1 = \int_0^{\Delta a_c} \sigma^\text{coh}_{12}(X_1) D(X_1) \, dX_1,
\]

which determines the constant \( \tau^\text{remote}_d \) in equation (12), as the portion of the stress intensity factor caused by \( \tau^\text{remote}_d \) can be solved directly by linear fracture mechanics theory. In equation (14), \( L \) is the distance from the crack tip to the specimen boundary, as illustrated in figure 2(a). We are assuming that

\[
L > L_R.
\]
Thus, according to hypothesis 7 in §2.1, equation (14) can be rewritten as

\[
\left( \frac{K_{\text{dyn}}}{E} \right)^2 = \frac{1}{2} \int_{-L_r}^{\Delta a_{\text{c}}} \tau_{\text{eq}}^{\text{remote}}(X_1) D(X_1) \, dX_1 \\
= \int_0^{\Delta a_{\text{c}}} \sigma_{12}^{\text{coh}}(X_1) D(X_1) \, dX_1.
\]  

(16)

The Weertman moving-dislocation equation (1), together with the proposed model in figure 1 (b) and the generalized BCSD superposition in figure 2, is referred to as the GBCSD model in this paper.

§3. The Solution Procedure

Substituting equations (6) and (12) into equation (11) and then into equation (1), we obtain

\[
\sigma_{12}(X_1) = (\tau_d^{\text{remote}} + \alpha_1) D(X_1) + \alpha_2 \int_{-\infty}^{\Delta a_{\text{c}}} \frac{D(w)}{X_1 - w} \, dw,
\]

(17)

where \( \sigma_{12} \) is defined as

\[
\sigma_{12}(X_1) = \begin{cases} 
\sigma_{12}^{\text{coh}}(X_1) - \tau_{\text{remote}}, & 0 \leq X_1 \leq \Delta a_{\text{c}}, \\
-\tau_{\text{remote}}, & 0 > X_1.
\end{cases}
\]

(18)

For the problem illustrated in figure 1 (b), equation (17) defines a Riemann–Hilbert problem along the line segment \(-\infty < X_1 \leq \Delta a_{\text{c}}\). It can be transformed into the following expression using the Muskhelishvili (1992) method and the algorithm introduced by Weertman (1996) (see appendix B):

\[
\Phi(z) = \frac{g(z)}{2\pi i} \int \frac{\sigma_{12}(w) \, dw}{L - z - w} + \Xi(z),
\]

(19a)

where

\[
\Phi(z) = \frac{1}{2\pi i} \int \frac{D(w)}{L - z - w} \, dw, \quad g(z) = (L_x - z)^{\omega_1}(\Delta a_{\text{c}} - z)^{\omega_1}(L_x - z)^{\omega_2 - \omega_1},
\]

(19b)

and

\[
\Xi(z) = \frac{C(L_x - z)^{\omega_1}}{(\Delta a_{\text{c}} - z)^{\omega_1}},
\]

(19c)

where \( C \) is a constant, \( \Xi(z) \) is the singular term of the solution, \( \omega_1 \) and \( \omega_2 \) are constant complexes and \( L_x \) is an undetermined constant. After some elementary operations, we obtain

\[
D(X_1) = \beta_1 \sigma_{12}(X_1) + \beta_2 (L_x + X_1)^{-\omega_1}(\Delta a_{\text{c}} - X_1)^{-\omega_2} \\
\times \int_{-\infty}^{\Delta a_{\text{c}}} \frac{\sigma_{12}(w) \, dw}{(L_x + w)^{-\omega_1}(\Delta a_{\text{c}} - w)^{-\omega_2} (w - X_1)}.
\]

(19d)

where

\[
\beta_1 = \frac{\hat{\alpha}_1}{\hat{\alpha}_1^2 + \pi \omega_2^2}, \quad \beta_2 = -\frac{\omega_2}{\hat{\alpha}_1^2 + \pi \omega_2^2}, \quad \hat{\alpha}_1 = \alpha_1 + \tau_d^{\text{remote}}.
\]

(20)

The detailed derivation of equation (19) is given in appendix B.
The following boundary conditions are required for a solution:

$$\int_{\Delta a_{c_1}}^{0} D(w) \, dw = \bar{\pi}_1,$$  

(21)

where $\bar{\pi}_1$ is a material constant, denoting the maximum separation at the end of a cohesive zone (critical crack-tip-opening displacement (CTOD)).

§4. **Problem I: A cohesive solution for a constant decohesion force**

4.1. **Governing equations**

Consider the degenerated cohesive law

$$\sigma_{12}^{\text{coh}}(X_1) = \begin{cases} \tau_{\text{th}}, & 0 \leq X_1 \leq \Delta a_C, \\ 0, & \text{otherwise}. \end{cases}$$

(10b)

Under this situation, equation (18) becomes

$$\sigma_{12}(X_1) = \begin{cases} \tau_{\text{th}} - \tau_{\text{remote}}^{\text{remote}}, & 0 \leq X_1 \leq \Delta a_C, \\ \tau_{\text{remote}}, & 0 > X_1. \end{cases}$$

(22)

Figure 3(a) illustrates the cohesive law (10b) whereas figure 3(b) shows the cohesive stress distribution (22) in the cohesive zone. It is noteworthy that according to equation (22) a jump in the crack surface traction exists at the crack tip ($X_1 = 0$). In static and quasistatic cases, similar problems have been studied by Weertman (1981) and Hurtado and Weertman (1993).

We solve this problem by splitting the corresponding dislocation density into two parts:

$$D(X_1) = D^{\text{coh}}(X_1) + D^{\text{remote}}(X_1).$$

(23)
Substituting this relation into equation (17) and then repeating the derivation through equations (17)–(19) by $L_x = -L_R$, the following two governing equations respectively can be obtained:

$$D^\text{remote}(X_1) = -\beta_1 \tau^\text{remote} + \beta_2 (L_R + X_1)^{-\alpha_1} (\Delta a_C - X_1)^{-\alpha_2}$$

$$\times \int_{-L_R}^{\Delta a_C} \frac{-\tau^\text{remote} d\omega}{(L_R + \omega)^{-\alpha_1} (\Delta a_C - \omega)^{-\alpha_2} (\omega - X_1)}$$

for $X_1 \leq \Delta a_C$ \hspace{1cm} (24a)

and

$$D^\text{coh}(X_1) = \beta_1 \tau_{th} + \beta_2 (L_R + X_1)^{-\alpha_1} (\Delta a_C - X_1)^{-\alpha_2}$$

$$\times \int_{0}^{\Delta a_C} \frac{\tau_{th} d\omega}{(L_R + \omega)^{-\alpha_1} (\Delta a_C - \omega)^{-\alpha_2} (\omega - X_1)}$$

for $0 \leq X_1 \leq \Delta a_C$, \hspace{1cm} (24b)

where a singular term such as equation (19c) has already been subtracted, and

$$\beta_1 = \frac{\hat{\alpha}_1}{\alpha_1^2 + \pi \alpha_2^2}, \quad \beta_2 = -\frac{\alpha_2}{\alpha_1^2 + \pi \alpha_2^2}, \quad \hat{\alpha}_1 = \alpha_1 + \tau^\text{remote}_d.$$

Equation (24a) defines the intersonic crack propagation under the crack surface traction:

$$\sigma_{12} = -\tau^\text{remote} - \tau^\text{remote}_d D^\text{remote}(X_1), \quad X_1 \leq \Delta a_C.$$ \hspace{1cm} (26a)

For this problem, the amplitude of the stress singularity at the crack tip is

$$K_{\text{II}}^{\text{dyn, remote}} = \frac{E}{2^{1/2}} \left( \int_{-\Delta a_C}^{\Delta a_C} \tau^\text{remote}(X_1) dX_1 - \int_{-L}^{\Delta a_C} \tau^\text{remote}_d (X_1) D^\text{coh}(X_1) dX_1 \right)^{1/2}. \hspace{1cm} (26b)$$

Equation (24b) defines the intersonic crack propagation under the crack surface traction:

$$\sigma_{12} = \begin{cases} \tau_{th} - \tau^\text{remote}_d D^\text{coh}(X_1), & 0 \leq X_1 \leq \Delta a_C, \\ -\tau^\text{remote}_d D^\text{coh}(X_1), & 0 > X_1, \end{cases} \hspace{1cm} (27a)$$

with

$$K_{\text{II}}^{\text{dyn, coh}} = \frac{E}{2^{1/2}} \left( \int_{0}^{\Delta a_C} \tau_{th} D^\text{coh}(X_1) dX_1 - \int_{-L}^{\Delta a_C} \tau^\text{remote}_d (X_1) \left[ D^\text{coh}(X_1) \right]^2 dX_1 \right)^{1/2}. \hspace{1cm} (27b)$$

The singularity amplitude expressed in equation (27b) must be equal to that in equation (26b), so that after the summation of these two solutions the singularities can cancel each other, that is

$$K_{\text{II}}^{\text{dyn, coh}} = K_{\text{II}}^{\text{dyn, remote}}.$$

This relation, together with equation (12), determines the unknown constants $\tau^\text{remote}$ and $\tau^\text{remote}_d$. 
4.2. Crack tip singularity variation with crack surface and remote load condition

By integrating equation (24a) its solution shows a singularity $\omega_1$ at $X_1 = \Delta a_C$ (cohesive zone front edge) and a singularity $\omega_2$ at $X_1 = -L_R$; similarly, the solution of equation (24b) has singularities $\bar{\omega}_1$ at $X_1 = \Delta a_C$ and $\bar{\omega}_2$ at $X_1 = -L_R$. The proposed GBCSD model requires the superposition of these two individual available solutions to cancel the singularities; so

\[ \bar{\omega}_1 = \omega_1, \quad \bar{\omega}_2 = \omega_2. \quad (29a) \]

According to the equations (B 6) and (B 9) of appendix B the eigenvalues $\omega_1$ and $\omega_2$ are written as

\[ \omega = \frac{1}{\pi} \tan^{-1} \left( \frac{\alpha_2}{\alpha_1 + \tau^{\text{remote}}_d} \right), \quad \omega = -\omega_1 = -\omega_2. \quad (29b) \]

The corresponding relations between $\omega$, $\tau^{\text{remote}}_d$ and propagation speed $V$ are plotted in figure 4(a). The analysis given, for example, by Freund (1979) indicates that the field solutions of both equation (24a) and equation (24b) present ‘radiated stress singularity’ except for

\[ \alpha_1 + \tau^{\text{remote}}_d = 0 \quad \text{and} \quad \omega = \frac{1}{2}. \quad (29c) \]

The solutions with ‘radiated stress singularity’ have no physical meaning; so they cannot be used in BCSD superposition, where equation (29c) defines the unique eigenvalue of $\omega$. From figure 4(a) we find that, when $\tau^{\text{remote}}_d = 0$, the propagation speed degenerates to the Eshelby speed, that is $V = c_s 2^{1/2}$. By increasing the equivalent crack surface traction $\tau^{\text{remote}}_d$, the Eshelby speed is ‘shifted’ into two bifurcated branches.

Figure 4(b) shows the relationship between $\tau^{\text{remote}}_d$ and the ‘shifted’ Eshelby speeds at $\omega = \frac{1}{2}$. Obviously, when $c_s \leq V \leq 2^{1/2} c_s$, the solution is unstable because the propagation speed is increased by reducing $\tau^{\text{remote}}_d$. Therefore, the speeds within the region $2^{1/2} c_s < V < c_1$ belong to the family of well-posed solutions for steady-state propagation.
4.3. Solution of case 1

After subtracting the singularity term and substituting equation (29c) into the governing equations (24a) and (24b), we obtain one equation, namely

\[
D(X_1) = \begin{cases}
\beta_1 (\tau_{th} - r_{remote}) + \beta_2 (L_R + X_1)^{\omega} (\Delta a_C - X_1)^{\nu} \\
\times \left( \int_0^{\Delta a_C} \frac{(\tau_{th} - r_{remote}) \, dw}{(L_R + w)^{\omega} ((\Delta a_C - w)^{\nu} (w - X_1))} - \int_0^{\Delta a_C} \frac{(\tau_{th} - r_{remote}) \, dw}{(L_R - X_1)^{\omega}} \right), \\
-\beta_1 r_{remote} + \beta_2 (L_R + X_1)^{\omega} (\Delta a_C - X_1)^{\nu} \\
\times \left( \int_0^{\Delta a_C} \frac{(\tau_{th} - r_{remote}) \, dw}{(L_R + w)^{\omega} ((\Delta a_C - w)^{\nu} (w - X_1))} - \int_0^{\Delta a_C} \frac{(\tau_{th} - r_{remote}) \, dw}{(L_R - X_1)^{\omega}} \right)
\end{cases}
\]

\[
0 \leq X_1 \leq \Delta a_C, \\
0 > X_1.
\]

(30)

Applying the transformation,

\[
x = \frac{X_1 + x_a}{x_c},
\]

where

\[
x_a = \frac{L_R - \Delta a_C}{2}, \quad x_c = \frac{L_R + \Delta a_C}{2}, \quad (31)
\]

equation (30) becomes

\[
D(x) = \begin{cases}
\beta_1 (\tau_{th} - r_{remote}) + \beta_2 (1 - x^2)^{1/2} \\
\times \left( \int_1^{x_a/x_c} \frac{(\tau_{th} - r_{remote}) \, dw}{(w - x)(1 - w^2)^{1/2}} - \int_{-1}^{x_a/x_c} \frac{r_{remote} \, dw}{(w - x)(1 - w^2)^{1/2}} \right), \\
-\beta_1 r_{remote} + \beta_2 (1 - x^2)^{1/2} \\
\times \left( \int_1^{x_a/x_c} \frac{(\tau_{th} - r_{remote}) \, dw}{(w - x)(1 - w^2)^{1/2}} - \int_{-1}^{x_a/x_c} \frac{r_{remote} \, dw}{(w - x)(1 - w^2)^{1/2}} \right),
\end{cases}
\]

\[
\frac{x_a}{x_c} \leq x \leq 1,
\]

(32a)

These kinds of equation have been solved by Weertman (1996) (see appendix D):

\[
D(x) = \begin{cases}
\beta_1 (\tau_{th} - r_{remote}) - \frac{\beta_2 \tau_{remote}}{2} \left[ \overline{Y}_0(x) + \overline{Y}_1(x) \right], \\
-\beta_1 r_{remote} - \frac{\beta_2 \tau_{remote}}{2} \left[ \overline{Y}_0(x) + \overline{Y}_1(x) \right],
\end{cases}
\]

(32b)

where

\[
\overline{Y}_0(x) = \ln \left[ \frac{(1 - x^2)^{1/2} + \left[ 1 - (x_a/x_c)^2 \right]^{1/2}}{(1 - x^2)^{1/2} - \left[ 1 - (x_a/x_c)^2 \right]^{1/2}} \right],
\]

\[
\overline{Y}_1(x) = \ln \left[ \frac{\left( x_a/x_c \right) (1 - x^2)^{1/2} + \left[ 1 - (x_a/x_c)^2 \right]^{1/2}}{\left( x_a/x_c \right) (1 - x^2)^{1/2} - \left[ 1 - (x_a/x_c)^2 \right]^{1/2}} \right].
\]
The crack-opening displacement (COD) can be obtained by integrating equation (32):

\[
\text{COD}(x) = 2\left[[u_1]\right] = -x_c \int_x^1 D(x) \, dx. \tag{33}
\]

In the cohesive zone (1 > x ≥ x_a/x_c),

\[
\text{COD}(x) = -x_c \beta (\tau_{th} - \tau_{\text{remote}})(1 - x) + \frac{\beta_2 \tau_{\text{remote}}}{2} \times \left[ \frac{\pi}{2} - \sin^{-1} x \right] (x_c^2 - x_a^2)^{1/2} - (x_a - x_c) \left[ \overline{Y}_0(x) + \overline{Y}_1(x) \right]. \tag{34}
\]

The cohesive zone size \( \Delta a_c \) is determined by this equation and equation (21).

Figure 5(a) shows the solution of the dislocation density \( D(x) \) for various values of \( \tau_{\text{d, remote}} \). This plot indicates that the amplitude of the dislocation density distribution is increased by increasing \( \tau_{\text{d, remote}} \), implying that a faster crack propagation speed results in a higher dislocation density; consequently, a higher crack surface separation (COD), as demonstrated in figure 5(b). At the crack tip, that is at \( x_1 = 0 \) or \( x = x_a/x_c \), in figure 5(a), \( D(x) \) presents a logarithm’s singularity owing to the terms \( \overline{Y}_0 \) and \( \overline{Y}_1 \) in equation (32). This is because the crack surface traction jumps from \( \tau_{th} - \tau_{\text{remote}} \) to \( \tau_{\text{remote}} \) in the model proposed in figure 2. However, the terms \( \overline{Y}_0 \) and \( \overline{Y}_1 \) cancel each other in the COD solution (34). Figure 5(c) shows the computed cohesive zone size \( \Delta a_c \) against the crack propagation speed when the critical CTOD \( \overline{u}_1 \) is given according to equations (21) and (34). Obviously, a larger \( \overline{u}_1 \) allows a longer cohesive zone. When \( \overline{u}_1 \) is fixed, the cohesive size is inversely proportional to crack propagation speed.

![Figure 5](image-url)
§ 5. PROBLEM II: SOLUTION FOR THE SEPARATION-DEPENDENT DECOHESION (EQUATION (10a))

5.1. Governing equations

Consider the traction–separation law defined by equation (10a):

$$
\sigma_{12}^{\text{coh}}(X_1) = \tau_{\text{th}} \left( 1 - \frac{\pi}{I} |u_1(X_1)| \right) \quad \text{for} \quad 0 \leq X_1 \leq \Delta a_c
$$

(10a)
This is applied in the GBCSD model as surface traction:

$$\sigma_{12}(X_1) = \begin{cases} \sigma_{12}^{\text{coh}}(X_1) - \tau_{\text{remote}}, & 0 \leq X_1 \leq \Delta a_C, \\ -\tau_{\text{remote}}, & 0 > X_1. \end{cases}$$ (18)

Figure 6 illustrates the problem described by equations (10a) and (18). In this case we still use the transformation (31); so the decohesion-induced separation is

$$[u_1(X_1)] = -\frac{1}{2} \int_{X_1}^{\Delta a_C} D(w) \, dw = -\frac{X_c}{2} \int_{X_c-x_a}^l D(w) \, dw,$$ (35)

where the coordinate $x$ is defined by equation (31).

At the crack tip ($X_1 = 0$), $\sigma_{12}^{\text{coh}}$ vanishes, so equation (10a) implies that the term $l/\tau_1$ has the value of half the CTOD. Since $\tau_1$ is constant, the length scale $l$ actually governs the amplitude of $[u_1(0)]$; thus, it scales the decohesion energy and, implicitly, determines the cohesive zone size $\Delta a_C$.

Similar to case I, we assume that the dislocation density function $D(x)$ can also be divided into two parts:

$$D(x) = D^I(x) + D^\prime(x),$$ (36)

where $D^I(x)$ denotes the solution (32), that is the dislocation density caused by remote boundary conditions and the constant cohesive traction $\tau_{\text{th}}$ in case I, whereas $D^\prime(x)$ is the portion due to the ‘separation-induced traction’, denoted as $-(\tau_{\text{th}} \tau_1 / l) [u_1(X_1)]$, in equation (10a). Again, the GBCSD model requires that each part of equation (36) is free of ‘stress singularity radiation’ and the superposition of all solutions cancels out the singularity at the cohesive zone tip. Therefore, equations (29c) are still suitable for this problem:

$$\alpha_1 + \tau_d^{\text{remote}} = 0 \quad \text{and} \quad \omega = \frac{1}{2}. \quad (29c)$$

Figure 6. Problem II: (a) traction–separation law; (b) distribution of traction in the cohesive zone.
5.2. Iteration solution

An iteration solution method is introduced in this section. The methodologies of numerical analysis solving ordinary and partial differential equations and integral equations have been given by Lowan (1957), Belytschko and Hughes (1983) and Belytschko et al. (2000).

We focus on the solution within the cohesive zone, that is \( x_1 \in [0, \Delta a_C] \) or, alternatively, \( x \in [x_c/x_a, 1] \) according to equation (31). Substituting equations (35) and (36) into the governing equation (19) and applying equations (29) and (31) and the solution (32), we obtain the governing equation for \( D^\mu(x) \):

\[
D^\mu(x) = \overline{p}_2 \left[ \overline{p}_1 \int_x^1 D(w) \, dw + q^{(\alpha, \omega)}(x) \right] - \left( \int_w^1 D(z) \, dz \right) \frac{q^{(\alpha, -\omega)}(w) \, dw}{w - x} \text{ for } x \in \left[ \frac{x_c}{x_a}, 1 \right],
\]

where

\[
\overline{p}_i = -\frac{\Delta a_C \pi_1}{l} \beta_i \quad \text{and} \quad \frac{x_c}{x_a} = \frac{L_R - \Delta a_C}{L_R + \Delta a_C}.
\]

The governing equation (37) can be rewritten in the form of a projection operation:

\[
D^\mu(x) = \lambda P(l, \Delta a_C, D(x)), \quad \lambda = \overline{p}_2.
\]

For given \( l, \Delta a_C \) and \( D^1 \), \( P \) projects function \( D \) on to \( D^\mu \). Obviously, \( P \) is a ‘linear projector’ which can be interpreted as follows: for a total dislocation density \( D_a \), equation (39) has a corresponding solution \( D_a^\mu \) and, for another total solution \( D_b \), it has \( D_b^\mu \); then

\[
\lambda P(l, \Delta a_C, D_a(x) + D_b(x)) = D_a^\mu(x) + D_b^\mu(x)
\]

and

\[
\lambda P(l, \Delta a_C, k D_a(x)) = k D_a^\mu(x).
\]

Also, according to the physical nature of the problem we know that \( D^\mu \), the projected solution of equation (39), has the same order of singularity as \( D^1 \); so the projection \( P \) does not bring extra singularity to the solution.

We first consider the case when the cohesive zone size \( \Delta a_C \) is fixed. A corresponding iteration scheme can be written as follows:

\[
\begin{align*}
\Delta D_0^\mu &= \lambda P(l, \Delta a_C, D^1), \quad D_0^\mu = \Delta D_0^\mu, \\
\Delta D_1^\mu &= \lambda P(l, \Delta a_C, \Delta D_0^\mu), \quad D_1^\mu = D_0^\mu + \Delta D_0^\mu, \\
\Delta D_2^\mu &= \lambda P(l, \Delta a_C, \Delta D_1^\mu), \quad D_2^\mu = D_1^\mu + \Delta D_1^\mu, \\
&\vdots \\
\Delta D_n^\mu &= \lambda P(l, \Delta a_C, \Delta D_{n-1}^\mu), \quad D_n^\mu = D_{n-1}^\mu + \Delta D_{n-1}^\mu, \\
\end{align*}
\]

Alternative expressions for equations (41) are

\[
\Delta D_n^\mu = \lambda P(l, \Delta a_C, \lambda P(l, \Delta a_C \ldots)) = \lambda^{n+1} P^{n+1}(l, \Delta a_C, D^1)
\]
\[ D^\mu = D_0^\mu + \lim_{n \to \infty} \left( \sum_{k=1}^{n} \Delta D_k^\mu \right) = \lim_{n \to \infty} \left( \sum_{k=0}^{n} \lambda^k P_k(l, \Delta a_C, D^1) \right). \] (42b)

We know that \( P \) does not bring extra singularity; so equation (42a) is a convergence if and only if \(|\lambda| < 1\).

For the problem dealt with in the paper, after the critical CTOD \( 2[\bar{\eta}_1(\Delta a_C)] \) is given, updating the dislocation density will alter the displacement distribution and, thus, adjust the cohesive zone size to match the boundary condition at the crack tip. Therefore, in the present work the following iteration procedure is introduced:

\[ D_0^\mu = \lambda P(l, \Delta a_0^C, D^1), \quad \Delta a_0^C = \Delta a_C(D^1), \quad \Delta D_0^\mu = D_0^\mu, \] (43a)

where \( \Delta a_0^C \) is the cohesive zone size corresponding to the dislocation density \( D(x) = D^1(x) \). Subsequently,

\[ D_0^\mu = \lambda P(l, \Delta a_0^C, D^1 + D_0^\mu), \quad \Delta a_1^C = \Delta a_C(D^1 + D_0^\mu), \quad \Delta D_1^\mu = D_1^\mu - D_0^\mu, \] (43b)

\[ \vdots \]

\[ D_n^\mu = \lambda P(l, \Delta a_1^C, D^1 + D_{n-1}^\mu), \quad \Delta a_C = \Delta a_C(D^1 + D_{n-1}^\mu), \quad \Delta D_n^\mu = D_n^\mu - D_{n-1}^\mu, \] (43c)

where \( \Delta a_n^C \) is the cohesive zone size corresponding to the dislocation density \( D^1 + D_{n-1}^\mu \) for the given \( 2[\bar{\eta}_1(\Delta a_C)] \). The convergence of this scheme can be stated as follows: for an arbitrary \( l \) with physically reasonable value, the solution obtained according to equations (43) is a convergence if and only if the solution series \( D_0^\mu, D_1^\mu, D_2^\mu, \ldots, D_n^\mu, \ldots \) forms a Cauchy sequence; that is, by giving any small quantity \( \varepsilon \), one can find an integer \( N \) that, for all \( n > N \), obeys

\[ \| \Delta D_n^\mu \|_L \leq \varepsilon \] (44)

where \( \| \Delta D_n^\mu \|_L \) is the \( L^2 \) norm defined by (Belytschko et al. 2000)

\[ \| \Delta D_n^\mu \|_L = \left[ \int_0^X (D_n^\mu - D_{n-1}^\mu)^2 \, dX \right]^{1/2}. \] (45)

### 5.3. Results and discussion

In the present analysis the coefficient \( \bar{\eta}_1 \) in equation (10a) is taken to be the unit. Thus, the length scale \( l \) actually specifies the value of the critical CTOD, denoted as \( [\bar{\eta}_1(0)] \). In order to compare with the solution obtained in case I, the ratio \( R_l \) given by

\[ R_l = \frac{[\bar{\eta}_1(0)](10a)}{[\bar{\eta}_1(0)](10b)} = \frac{l}{[\bar{\eta}_1(0)](10b)} \] (46)

is introduced to characterize the effects of the length scale \( l \) and the separation-induced traction, that is the term \( (\tau_{th} \bar{\eta}_1/l)[\bar{u}_1(X_1)] \) in equation (10a).

Figure 7(a) displays the \( \Delta D_n^\mu \) values computed using equation (43), for the iteration number \( n = 0, 1, 2, 3, 5 \) and 11 when \( R_l = 0.9 \). According to the transformation (31), in this diagram the right-hand end of the horizontal axis is the front edge of the cohesive zone, whereas the starting point of each curve represents the crack tip.
position corresponding to $\Delta \bar{a}_C$. Thus, the magnitude of $\Delta \bar{a}_C$ decreases away from the size in the case I solution when the iteration number increases. The zeroth-order solution $\Delta D_0^n$ presents a logarithm singularity at its left-hand end, which cancels the singularity in the solution (32) that is shown in Figure 5(a). After ten iterations, $\Delta D_n^n$ ($n = 11$) is almost a straight line on the zero level. In figure 7(b), $\|\Delta D_n^n\|_L$ is plotted for various $R_i$, providing quantitative measurement of the convergence rate of the introduced scheme. It demonstrates that $\|\Delta D_n^n\|_L$ is approximately linearly proportional to $\exp(-\text{iteration number})$. After 20 iterations, the values of $\|\Delta D_n^n\|_L$ are less than $10^{-10}$ for all cases.

Figure 8 shows the solutions in this case. Figure 8(a) is the distribution of the traction $\sigma_{12}$ in the cohesive zone, where the horizontal coordinate is normalized by the cohesive zone size. No significant differences can be seen when $R_i$ varies. However, these curves exhibit a highly nonlinear character, although a linear relationship between force and separation occurs in equation (10a). Figure 8(b) depicts the corresponding distributions of the dislocation densities on the horizontal axis normalized by the cohesive zone size, where smaller $R_i$ leads to lower amplitude
of $D(x)$. This phenomenon can be explained by figure 8(c) where the corresponding separations are plotted on the horizontal axis which is not normalized by the cohesive zone size. It demonstrates that a smaller $R_l$ leads to a shorter cohesive zone size. Consequently, the corresponding dislocation density is also lower, as shown in figure 8(b). Figure 8(d) shows explicitly the relationship between $R_l$ and $A_C$, the relative cohesive zone size defined by

$$A_C = \frac{\Delta a_C(R_l)}{\Delta a_C \text{(case I)}}.$$  

$$\text{(47)}$$

Figure 8. The solutions of case II for various length scales $R_l$: (a) distributions of traction in the cohesive zone; (b) distributions of dislocation density; (c) separations; (d) cohesive zone size against length scale.
§ 6. Problem iii: solution for the traction–separation law (equation (6))

6.1. Governing equations

In this problem the decohesion law

\[
\sigma_{12}^{\text{coh}}(X_1) = \tau_{th} \left[ 1 - \frac{\pi_1}{l} [u_1(X_1)] \right] + \left( \pi_2^a \frac{l}{h} + \pi_2^b \frac{mV}{h} \right) D(X_1) \]

is applied in the GBCSD model as the surface traction

\[
\sigma_{12}(X_1) = \begin{cases} 
\sigma_{12}^{\text{coh}}(X_1) - \tau_{\text{remote}}, & 0 \leq X_1 \leq \Delta a_C, \\
-\tau_{\text{remote}}, & 0 > X_1.
\end{cases}
\]

Figure 9 illustrates the problem described by equations (6) and (18).
Again we focus on the solution within the cohesive zone, that is $X_1 \in [0, \Delta a_C]$. Applying the transformation

$$x = \frac{2X_1}{\Delta a_C} - 1$$

(48)

at $X_1 = 0 \rightarrow x = -1$ and $X_1 = \Delta a_C \rightarrow x = 1$,

$$X_1^{\omega_1}(\Delta a_C - X_1)^{\omega_2} = \left(\frac{\Delta a_C}{2}\right)^{\omega_1+\omega_2}(1+x)^{\omega_1}(1-x)^{\omega_2} = \left(\frac{\Delta a_C}{2}\right)^{\omega_1+\omega_2} q^{\omega_1+\omega_2}(x).$$

(49)

Also

$$[u_1(X_1)] = -\frac{1}{2} \int_{X_1}^{\Delta a_C} D(X_1) \, dX_1 = -\Delta a_C \int_x^1 D(x) \, dx \quad \text{for} \quad 0 \leq X_1 \leq \Delta a_C.$$  

(50)

After the transformation (48), the governing equation (19d) becomes

$$D(X_1) = \beta_1 \sigma_{12}(x) + \beta_2 q^{(\omega_1, \omega_2)}(x) \int_{-\infty}^1 \frac{\sigma_{12}(w) \, dw}{q^{(\omega_1, \omega_2)}(w)(w-x)}.$$  

(51)

Similar to cases I and II, the condition to remove the singularities at $X_1 = 0, X_1 = \Delta a_C$ ($x = \pm 1$) requires

$$\text{Re}(\omega_1) = \text{Re}(\omega_2) = \omega, \quad \text{Im}(\omega_1) + \text{Im}(\omega_2) = 0.$$  

(52)

However, as the additional terms involved $D(x)$ are introduced into equation (51), equation (29c) is no longer available.

Figure 9. Problem III: (a) traction–separation law; (b) distribution of traction in the cohesive zone.
Substituting equations (6), (18), (50) and (52) into equation (51), the governing equation, for \(|x| \leq 1\), becomes

\[
q^{(\alpha_1, \alpha_2)}(x)\beta_2 \int_{-\infty}^{-1} \frac{dw}{q^{(\alpha_1, \alpha_2)}(w)(w-x)} = \beta_1 (\tau_{th} - \tau_{\text{remote}}) + q^{(\alpha_1, \alpha_2)}(x)\beta_2 \int_{-1}^{1} \frac{dw}{q^{(\alpha_1, \alpha_2)}(w)(w-x)}
\]

\[
+ (\tilde{\beta}_1 - 1)D(x) + q^{(\alpha_1, \alpha_2)}(x)\tilde{\beta}_2 \int_{-1}^{1} D(w) \frac{dw}{q^{(\alpha_1, \alpha_2)}(w)(w-x)}
\]

\[
+ \tilde{\beta}_1 \int_{x}^{1} D(w) dw + q^{(\alpha_1, \alpha_2)}(x)\tilde{\beta}_2 \int_{-1}^{1} \left( \int_{w}^{1} D(z)dz \right) \frac{dw}{q^{(\alpha_1, \alpha_2)}(w)(w-x)},
\]

where

\[
\tilde{\beta}_i = -\frac{\Delta a_c \pi_1}{l} \beta_i, \quad \tilde{\beta}_i = \left( \pi_2^2 \frac{l}{h} + \pi_2^2 \frac{mV}{h} \right) \beta_i, \quad i = 1, 2.
\]

6.2. The expansion of the Jacobi polynomials

We apply the Jacobi polynomial series \(P_n^{\alpha_1, \alpha_2}(x)\) (Szegö 1939) to represent the dislocation density \(D^n(x)\) in equation (53):

\[
D^n(x) = \sum_{n=1}^{\infty} \left[ \kappa_n P_n^{(-\alpha_1, -\alpha_2)}(x) + \bar{\pi}_n \bar{P}_n^{(-\alpha_1, -\alpha_2)}(x) \right],
\]

where \(\kappa_n\) are the coefficients to be determined, and \(\pi_n\) and \(\bar{\pi}_n^{\alpha_1, \alpha_2}(x)\) represent the conjugate of \(\kappa_n\) and \(P_n^{\alpha_1, \alpha_2}(x)\) respectively. The \(n\)th-order Jacobi polynomial \(P_n^{\alpha_1, \alpha_2}(x)\) is defined by

\[
P_n^{(\alpha_1, \alpha_2)}(x) = \frac{(-)^n}{2^n n! q^{(\alpha_1, \alpha_2)}(x)} \frac{d^n}{dx^n} \left[ q^{(\alpha_1, \alpha_2)}(x)(1-x^2)^n \right].
\]

Obviously

\[
P_0^{(\alpha_1, \alpha_2)}(x) \equiv 1.
\]

The Jacobi polynomials form an orthogonal complete function basis weighted by \(q^{(\alpha_1, \alpha_2)}(x)\):

\[
\int_{-1}^{1} q^{(\alpha_1, \alpha_2)}(x) P_m^{(\alpha_1, \alpha_2)}(x) P_n^{(\alpha_1, \alpha_2)}(x) dx = C_n^{(\alpha_1, \alpha_2)} \delta_{mn},
\]

where

\[
C_n^{(\alpha_1, \alpha_2)} = \frac{1}{2n + 1 + \omega_1 + \omega_2} \frac{\Gamma(n + 1 + \omega_1) \Gamma(n + 1 + \omega_2)}{n! \Gamma(n + 1 + \omega_1 + \omega_2)}.
\]

The following relations of the Jacobi polynomials (Szegö 1939, Bateman 1953) are useful in the present analysis: firstly,

\[
\int_{x}^{1} P_n^{(\alpha_1, \alpha_2)}(t) dt = \frac{2}{n + \omega_1 + \omega_2} \left[ P_n^{(\alpha_1 - 1, \alpha_2 - 1)}(1) - P_n^{(\alpha_1 - 1, \alpha_2 - 1)}(x) \right],
\]
secondly,
\[
\frac{\pi P_n^{(\omega_1, \omega_2)}(x)}{\sin(\pi \omega_1)} + q^{(\omega_1, \omega_2)}(x) \int_{-1}^{1} \frac{dt}{x-t} \frac{P_n^{(-\omega_1, -\omega_2)}(t)}{q^{(\omega_1, \omega_2)}(t)}(t) = s^{(\omega_1, \omega_2)}(\omega_1, \omega_2) \beta^{(\omega_1, \omega_2)}(\omega_1, \omega_2)(x),
\]
where the coefficients are given in appendix C; thirdly,
\[
-\frac{\pi P_n^{(-\omega_1-1, -\omega_2-1)}(x)}{\sin(\pi \omega_1)} + q^{(\omega_1, \omega_2)}(x) \int_{-1}^{1} \frac{dt}{x-t} q^{(-\omega_1-1, -\omega_2-1)}(t) \beta^{(-\omega_1-1, -\omega_2-1)}(t) = \sum_{j=0}^{\infty} \lambda_j q^{(\omega_1, \omega_2)}(x) \beta^{(-\omega_1-1, -\omega_2-1)}(x),
\]
where the coefficients are given in appendix C; fourthly,
\[
\int_{-1}^{1} q^{(\omega_1, \omega_2)}(z) \beta^{(\omega_1, \omega_2)}(z) dz = q^{(1+\omega_1, 1+\omega_2)}(x) \beta^{(1+\omega_1, 1+\omega_2)}(x);
\]
fifthly, the Fourier–Jacobi expansion where, for an \(L^p(p \leq 2)\) integrable function \(f(x)\) within \(-1 \leq x \leq 1\), \(f(x)\) can be expressed as the infinite series spanned by Jacobi polynomials (Szegö 1939), that is
\[
f(x) = \sum_{n=0}^{\infty} f_n P_n^{(\omega_1, \omega_2)}(x), \quad f_n = C_n^{(\omega_1, \omega_2)} \int_{-1}^{1} f(x) q^{(\omega_1, \omega_2)}(x) \beta^{(\omega_1, \omega_2)}(x) dx.
\]

6.3. The solution procedure of equation (53)

Substituting the real part of equation (55) into equation (53), the problem stated in §6.3 is transformed into a problem to find the solutions for each singular integral equation in the three groups
\[
f_0(x) = k_{11} \psi_1(x) + k_{12} \psi_2(x) \int_{-1}^{1} \psi_{n}(w) \frac{dw}{q^{(\omega_1, \omega_2)}(w)(w-x)}, \quad i = 1, 2, 3, \quad n \geq 0:
\]
for \(i = 1\),
\[
\psi_{n1}(x) = \tau_{th} - \tau_{\text{remote}}, \quad k_{11} = \beta_1, \quad k_{12} = \beta_2;
\]
\[
f_1(x) = \beta_2 \psi_2(x) \int_{-1}^{1} \frac{\tau_{\text{remote}} dw}{q^{(\omega_1, \omega_2)}(w)(w-x)}, \quad n = 0;
\]
for \(i = 2\),
\[
\psi_{n2}(x) = P_n^{(-\omega_1, -\omega_2)}(x), \quad k_{21} = \tilde{\beta}_1 - 1, \quad k_{22} = \tilde{\beta}_2;
\]
\[
f_{n2}(x) = f_n(x), \quad k \geq 0, \quad 0 \leq n < \infty;
\]
for \(i = 3\),
\[
\psi_{n3}(x) = \int_{x}^{1} P_n^{(-\omega_1, -\omega_2)}(w) dw, \quad k_{31} = \tilde{\beta}_1, \quad k_{32} = \tilde{\beta}_2, \quad f_{n3}(x) = f_{n3}(x), \quad k \geq 0, \quad 0 \leq n < \infty.
\]
The linear singular integral equations (63a)–(63d) can also be written in the form of the projection operation

\[ f_{ni}(x) = \rho_i \Phi(k, \omega_1, \omega_2, \psi_{ni}(x)) \quad \text{for } i = 1, 2, 3, \quad \rho_1 = \beta_2, \quad \rho_2 = \beta_2, \quad \rho_3 = \beta_2. \]  

(64)

Based on equations (63a)–(63d) and (64), we obtain an alternative expression for the governing equation (53):

\[ \sum_{n=0}^{\infty} [f_{n2}(x) + f_{n3}(x)] = f_{01}(x) - \beta_2 \Phi(\beta_1, \omega_1, \omega_2, \tau_{th} - \tau_{remote}). \]  

(65)

When \( i = 1 \) in equation (63), it is actually the problem studied in case I in the previous section. Now we study the solution for \( i = 3 \). By applying equation (58) to the \( \psi_{n3}(x) \) in equation (63c), its governing equation (63) becomes

\[ f_{n3}(x) = \frac{2\beta_2}{n - \omega_1 - \omega_2} \times \left[ \Phi \left( \frac{\beta_1}{\beta_2}, \omega_1, \omega_2, \Phi^{(-\omega_1-1,-\omega_2-1)}_{n+1}(1) \right) - \Phi \left( \frac{\beta_1}{\beta_2}, \omega_1, \omega_2, \Phi^{(-\omega_1-1,-\omega_2-1)}_{n+1}(x) \right) \right]. \]  

(66)

Using the transformation introduced in equation (C3) of appendix C, the second term on the right-hand side of equation (64) can be rewritten as

\[ \Phi \left( k_3, \omega_1, \omega_2, \Phi^{(-\omega_1-1,-\omega_2-1)}_{n+1}(w) \right) = \Phi \left( k_3, \omega_1 + 1, \omega_2 + 1, \Phi^{(-\omega_1-1,-\omega_2-1)}_{n+1}(w) \right). \]  

(67)

Substituting equation (67) into equation (66) and applying equation (60), it becomes

\[ f_{n3}(x) - \frac{2\beta_2}{n - \omega_1 - \omega_2} \Phi \left( \frac{\beta_1}{\beta_2}, -\omega_1 - 1, -\omega_2 - 1, \Phi^{(-\omega_1-1,-\omega_2-1)}_{n+1}(1) \right) \]

\[ = - \frac{2\beta_2}{n - \omega_1 - \omega_2} \left( \sum_{j=0}^{2} \gamma_j q_{n,N}(\omega_1, \omega_2) \Phi^{(\omega_1, \omega_2)}_{n-J}(x) \right) \]  

(68)

if the following relation is satisfied:

\[ \frac{\beta_1}{\beta_2} = \frac{\pi}{\sin[\pi(\omega_1 + 1)]}. \]  

(69)

For the case when \( i = 2 \), by applying equation (59) to the governing equations (63a)–(63c), we obtain

\[ f_{n2}(x) = s^{(n, \omega_1, \omega_2)} q^{(\omega_1, \omega_2)} \Phi^{(\omega_1, \omega_2)}_{n+1}(x) \]  

(70)

if the following relation is satisfied:

\[ \frac{\beta_1 - 1}{\beta_2} = \frac{\pi}{\sin(\omega_1 \pi)}. \]  

(71)

### 6.4. Results and discussion

Equations (69) and (71) are the conditions to determine the eigenvalues \( \omega_1 \) and \( \omega_2 \) and the crack propagation velocity. The solutions are

\[ \text{Re}(\omega_1) = \frac{1}{2}, \quad \text{Im}(\omega_1) = -\log \left[ \left( \frac{\beta_2^2 \pi^2}{\beta_1^2} - 1 \right)^{1/2} - \frac{\beta_2 \pi}{\beta_1} \right] \]  

(72)
and
\[ \frac{\hat{\beta}_1 - 1}{\beta_2} = -\frac{\tilde{\beta}_1}{\tilde{\beta}_2}. \]  

(73)

Associated with equation (52), equation (72) determines the eigenvalues \( \omega_1 \) and \( \omega_2 \); then, the imaginary part of the Jacobi weight function vanishes, that is
\[ q^{(\omega_1, \omega_2)}(x) = \Re \left( q^{(\omega_1, \omega_2)}(x) \right). \]  

(74)

The governing equation (63) becomes
\[ \sum_{n=0}^{\infty} \kappa_n q^{(\omega_1, \omega_2)}(x) \left( \tilde{\beta}_2 \tilde{\rho} (n, \omega_1, \omega_2) \tilde{p}^{(\omega_1, \omega_2)}(x) - \frac{2 \tilde{\beta}_2}{n - \omega_1 - \omega_2} \sum_{j=0}^{2} \lambda_j \tilde{p}_{n-j}^{(\omega_1, \omega_2)}(x) \right) \]
\[ + \sum_{n=0}^{\infty} \frac{2 \tilde{\beta}_2 \kappa_n \tilde{p}_{n+1}^{(\omega_1, \omega_2)}(1)}{n - \omega_1 - \omega_2} = f_0(x) - \beta_2 \mathcal{P}(\beta_1, \omega_1, \omega_2, \psi_0(x)). \]  

(75)

The right-hand side of equation (75) equals the dislocation density function solved in case I, which can be expanded into a Jacobi series by using equation (62):
\[ D_{\text{case I}}(x) = q^{(\omega_1, \omega_2)}(x) \frac{D_{\text{case I}}(x)}{q^{(\omega_1, \omega_2)}(x)} \]
\[ = \sum_{n=0}^{\infty} d_n q^{(\omega_1, \omega_2)}(x) \tilde{p}_n^{(\omega_1, \omega_2)}(x), \]  

(76)

\[ d_n = \int_{-1}^{1} D_{\text{case I}}(x) \tilde{p}_n^{(\omega_1, \omega_2)}(x) \, dx. \]

Substituting equation (76) back into equation (75) and comparing the coefficients for \( \tilde{p}_n^{(\omega_1, \omega_2)} \), we obtain the following recurrence formula for the coefficient \( \kappa_n \):
\[ \kappa_0 = 0, \]
\[ \kappa_1 = 0, \]
\[ \kappa_2 = -\frac{(2 - \omega_1 - \omega_2) d_0}{2 \tilde{\beta}_2^2}, \]
\[ \kappa_3 = -\frac{d_1}{2 \tilde{\beta}_2} \left( \frac{2 - \omega_1 - \omega_2}{\lambda_1^2} + \frac{3 - \omega_1 - \omega_2}{\lambda_2^2} \right), \]
\[ \kappa_4 = -\frac{d_2}{2 \tilde{\beta}_2} \left( \sum_{j=0}^{2} \frac{4 - j - \omega_1 - \omega_2}{\lambda_{4-j}^2} \right)^{-1}, \]
\[ \kappa_4 = d_3 \left( \frac{\xi^{(\omega_1, \omega_2)}}{\beta_2} - \frac{1}{2 \tilde{\beta}_2} \sum_{j=0}^{2} \frac{5 - j - \omega_1 - \omega_2}{\lambda_{5-j}^2} \right)^{-1}, \]
\[ \kappa_n = d_{n-2} \left( \frac{\xi^{(n-3, \omega_1, \omega_2)}}{\beta_2} - \frac{1}{2 \tilde{\beta}_2} \sum_{j=0}^{2} \frac{n - j - \omega_1 - \omega_2}{\lambda_{n-j}^2} \right)^{-1} \quad \text{for } n \geq 5. \]  

(77)
The eigensolution of equation (73) can be written in the form

\[ \hat{\alpha}_1 = \frac{h}{l \pi_2^2 + m V \pi_2^2 (\hat{\alpha}_1^2 + \pi \alpha_2^2)}, \]  

(78)

where \( \hat{\alpha}_1 \) and \( \alpha_2 \) are given by equations (2) and (25) respectively. This equation provides the relationship between the crack propagation speed, the material intrinsic length scale \( l \), the material extrinsic length scales \( h \), the strain-rate-hardening exponent \( m \) and the load parameter \( \tau_{\text{remote}} \). Figure 10 is a plot of the computed speed (normalized by shear wave speed) against \( \tau_{\text{remote}} \) for different values of the ratio \( h/l \). This figure is quite similar to the curve presented in figure 4(b) in §4. However, the ratio \( h/l \) shifts curves towards a higher speed. For each curve, the part right from the minimum saddle point gives a stable solution. The saddle point of each curve predicts the minimum intersonic propagation speed and corresponding remote force \( \tau_d^{\text{remote}} \) when these material constants are given. In figure 11 the ratio \( h/l \) is taken as the perpendicular axis, and the speed as the horizontal axis; the solved speed at the saddle points in figure 10 are plotted for various strain-rate-hardening exponents \( m \). This figure demonstrates, that by increasing the strain-rate-hardening exponent \( m \), the crack propagation speed will increase when the ratio \( h/l \) is fixed. From this diagram, one also sees that, when the material intrinsic length \( l \) is large or the width \( h \) of the weak path is small, the speed \( V \) approaches to \( 2^{1/2} c_s \). This can be explained according to equation (78). When \( h \) approaches zero or \( l \) approaches infinity, equation (78) degenerates to equation (29c) and the curves in figure 10 are convergent to the curve in figure 4(b) where the minimum intersonic propagation speed is \( 2^{1/2} c_s \) and no \( \tau_d^{\text{remote}} \) is required.

Considering the extreme case when \( h \) approaches \( b \), the Burgers tensor, the weak path allows dislocations to pass through one by one, whereas according to

![Figure 10](1093)

Figure 10. Relationship between the solved propagation velocity and \( \tau_d^{\text{remote}} \).
equation (7) we know that \( l \) is of the order of microns. Thus the ratio \( h/l \) is of the order of \( 10^{-3} \) and from figure 11 it can be seen that the speed should be quite close to \( 2^{1/2}c_s \). This coincides with the results of the pioneering dislocation researchers (Eshelby 1949, Frank 1949, Weertman et al. 1969) as they predicted that \( 2^{1/2}c_s \) is the favourite speed for intersonic dislocation motion. On the other hand, when \( l \) approaches infinity, according to the analysis of case II, it is required that the critical CTOD is infinite too. This is impossible for a steady-state growing crack but it can happen for smeared-out intersonic motion, as solved by Eshelby (1949), Frank (1949) and Weertman et al. (1969).

From figure 10, another conclusion can be drawn: for a weak path with finite \( l \) and \( h \), an intersonic shear failure cannot happen if \( \tau_d^{\text{remote}} = 0 \); that is, no crack surface loading or remote stress rate boundary condition is imposed. According to equation (12) the term \( \tau_d^{\text{remote}}D(X_t) \) represents a constant crack surface force in the moving coordinate originating at the moving crack tip. Hence, the boundary condition associated with \( \tau_d^{\text{remote}} \) can be interpreted as a point load or a distributed force with a constant distance to the moving crack tip. This boundary condition is necessary for steady-state intersonic crack growth.

To verify the analytical solution, several numerical simulations have been performed on the basis of the California Institute of Technology mode II intersonic crack propagation test (Rosakis et al. 1999) but a symmetric impact is imposed with a constant strain-rate-hardening power \( (m = 0.01) \). The computation is carried using the mesh-free method introduced by Liu and Chen (1995), Liu et al.
§7. Conclusions

(i) A model of dynamic mode II crack growth along a preferred weak path, which we refer to as GBCSD model, has been proposed to solve the dynamic crack propagation problem. In this model, intersonic crack propagation is considered as the accumulation of moving dislocations that obey the smeared-out moving-dislocation governing solution. An equivalent crack surface traction is introduced to represent the crack surface load and remote boundary condition including an imposed stress-rate boundary condition. A cohesive zone occurs ahead of the crack tip. BCSD superposition is applied in the dynamic analysis. Solutions for three types of traction–separation law, namely

(1) constant traction
(2) traction linearly dependent of separation
(3) strain gradient theory-based traction–separation laws

have been obtained using direct integration, iteration methods and the Jacobi polynomial expansion, respectively.

(ii) The solutions demonstrate that the intersonic speed region can be divided into two subdomains: the unstable domain \((c_s < V < V_{saddle})\) and the stable steady-state crack propagation domain \((V_{saddle} < V < c_d)\), where the speed that distinguishes these two subdomains is denoted \(V_{saddle}\). When a constant traction or the traction linearly dependent on separation is applied to a cohesive zone, \(V_{saddle} = 2^{1/2}c_s\); when the strain-gradient-based traction–separation law is applied, \(V_{saddle}\) is shifted by varying the remote load, the ratio \(h/l\) of length scales and the strain-rate-hardening exponent.

(iii) The propagation speed for intersonic model II crack propagation is found when the remote load and the decohesion law are given. This solution demonstrates the following properties.

(1) At the stable propagation domain, the propagation speed is increased by increasing the ratio \(h/l\).

(2) For the strain-rate-sensitive material (with larger strain-rate-hardening exponents), the propagation speed is higher than for those with smaller strain-rate-hardening exponents.

(3) In the case when the width of the weak preferred propagation path approaches the order of the single dislocation, the solution speed approaches \(2^{1/2}c_s\), the Eshelby speed.

(iv) For a weak path with a finite width and the decohesion law governed by the length scale \(l\) with a finite value, mode II intersonic shear failure will not take place if only a constant remote stress is imposed. The boundary condition such as a point force or a distributed load with a constant distance to the moving crack tip is necessary for steady-state intersonic crack propagation.
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APPENDIX A

A traction–separation law based on strain gradient theory has been derived by Kubair and Geubelle (2002):

\[ \sigma_{12}(X_1) = \tau_{th} \tilde{T}([u_1], l) \left( f^2 \left( \frac{\varepsilon_{\text{hom}}}{\varepsilon_0} \right) + ln \right)^{1/2} \left( 1 + \frac{\dot{\varepsilon}}{\varepsilon_0} \right)^m, \]  \hspace{1cm} (A1)

where \( \varepsilon_{\text{hom}} \) represents the average equivalent plastic strain that takes the value of the unit in this analysis, \( \eta \) is the equivalent strain gradient and \( l \) is the material intrinsic length defined by (Gao et al. 1999b,c)

\[ \eta = \left( \frac{1}{4} u_{i,jk} u_{i,jk} \right)^{1/2}, \quad l = 3 \frac{E_b}{\sigma_y}, \]

where \( b \) is the magnitude of the Burgers vector.

Then, the terms on the right-hand side of equation (A 1) can be linearized as

\[ \tilde{T}([u_1], l) = 1 - \frac{l_0}{\bar{u}} [u_1] \frac{1}{l} + lT_2 \left[ \frac{[u_1]^2}{l^2} \right] + l^2 T_3 \left[ \frac{[u_1]^3}{l^3} \right] + \ldots, \]  \hspace{1cm} (A2)

\[ (1 + ln)^{1/2} = 1 + \frac{1}{2} ln - \frac{1}{8} l^2 n^2 + \ldots, \]  \hspace{1cm} (A3)

\[ \left( 1 + \frac{\dot{\varepsilon}}{\varepsilon_0} \right)^m = 1 + m \frac{\dot{\varepsilon}}{\varepsilon_0} + \frac{m^2}{2} \left( \frac{\dot{\varepsilon}}{\varepsilon_0} \right)^2 + \ldots, \]  \hspace{1cm} (A4)

where \( l_0, \bar{u}, T_1, m \) and \( \varepsilon_0 \) are constants. We assume a uniform distribution of the dislocation along the height direction of the strip in figure 1(b). Hence

\[ \bar{\varepsilon} = \left( \frac{2}{3} \right)^{1/2} \left[ \frac{[u_1(X_1)]}{h} + O \left( \frac{[u_1(X_1)]}{h} \right) \right], \quad \bar{\varepsilon} = V \left( \frac{1}{6} \right)^{1/2} \left[ \frac{D(X_1)}{h} + O \left( \frac{D(X_1)}{h} \right) \right], \]  \hspace{1cm} (A5)

\[ \eta = \frac{1}{8^{1/2}} \left[ \frac{D(X_1)}{h} + O \left( \frac{D(X_1)}{h} \right) \right]. \]  \hspace{1cm} (A6)

Substituting equations (A2)–(A6) into equation (A 1) and leaving out the higher-order terms, we obtain

\[ \sigma_{12}(X_1) = \tau_{th} \left[ 1 - \frac{\pi_1}{\bar{u}} [u_1(X_1)] + \left( \pi_2 \frac{l}{h} + \pi_3 \frac{mV}{h} \right) D(X_1) \right], \]  \hspace{1cm} (A7)

where

\[ \pi_1 = \frac{l_0}{\bar{u}_1}, \quad \pi_2 = \frac{1}{8^{1/2}}, \quad \pi_3 = \frac{1}{\varepsilon_0} \left( \frac{1}{6} \right)^{1/2}, \]

where \( \tau_{th}, l_0, \varepsilon_0 \) and \( m \) are material constants.
APPENDIX B

§ B1. Derivations of equations (17)–(19)

The equation

\[ \sigma_{12}(X_1) = (\tau_d \text{remote} + \alpha_1)D(X_1) + \alpha_2 \int_{L} \frac{D(w)}{X_1 - w} \, dw \]  

(17)

defines a Riemann–Hillbert problem along the line segment \( L: -\infty < X_1 \leq \Delta a_c; X_2 = 0 \). When \( L \) is the sum of several closed contours, the solution procedure has been given by Muskhelishvili (1992). When \( L \) degenerates to a straight line, several cases of equation (14) have been studied by Weertman (1996).

We define the sectional holomorphic function in the complex plane \( z = X_1 + iX_2 \) by

\[ \Phi(z) = \frac{1}{2\pi i} \int_{L} \frac{D(w)}{z - w} \, dw \]  

(B1)

and let \( L \) have the direction from \( X_1 = -a_c \) to \( X_1 = 0 \). When \( z \) approaches \( X_1 \) on \( L \) from the right-hand side or left-hand side of \( L \), written as \( \Phi^+(X_1) \) or \( \Phi^-(X_1) \) respectively, the Plemelj formulæ (Muskhelishvili 1992) are available:

\[ \Phi^+(X_1) = \frac{1}{2} D(X_1) + \frac{1}{2\pi i} \int_{L} \frac{D(w)}{X_1 - w} \, dw, \]
\[ \Phi^-(X_1) = -\frac{1}{2} D(X_1) + \frac{1}{2\pi i} \int_{L} \frac{D(w)}{X_1 - w} \, dw. \]  

(B2)

Then, by substituting equation (B2) into equation (17) we obtain

\[ \sigma_{12}(X_1) = (\tau_d \text{remote} + \alpha_1)[\Phi^+(X_1) - \Phi^-(X_1)] + i\alpha_2 \pi [\Phi^+(X_1) + \Phi^-(X_1)] \]  

(B3)

or

\[ \sigma_{12}(X_1) = (\tau_d \text{remote} + \alpha_1 + i\alpha_2 \pi)\Phi^+(X_1) - (\tau_d \text{remote} + \alpha_1 - i\alpha_2 \pi)\Phi^-(X_1). \]  

(B4)

An alternative expression for equation (B4) is

\[ \Phi^+(X_1) = G(X_1)\Phi^-(X_1) + \frac{\sigma_{12}(X_1)}{\tau_d \text{remote} + \alpha_1 + i\alpha_2 \pi}, \]  

(B5)

where

\[ G(X_1) = \frac{\tau_d \text{remote} + \alpha_1 - i\alpha_2 \pi}{\tau_d \text{remote} + \alpha_1 + i\alpha_2 \pi} = \frac{A - iB}{A + iB} = \exp \left[ -2i \tan^{-1} \left( \frac{B}{A} \right) \right]. \]  

(B6)

On the other hand, the general solution of the Hilbert problem expressed as equation (B5) is (Muskhelishvili 1992)

\[ \Phi(X_1) = g(X_1)z(X_1), \]  

(B7)

where \( g(X_1) \) is the function to be determined and \( z(X_1) \) is an arbitrary polynomial. Hence

\[ \Phi^+(X_1) = \frac{g^+(X_1)}{g^-(X_1)} \Phi^-(X_1) \quad \text{or} \quad G(X_1) = \frac{g^+(X_1)}{g^-(X_1)}. \]  

(B8)
For the line segment $L$ defined as $L : -L_x < X_1 \leq \Delta a_C$

$$g(z) = (\Delta a_C - z)\omega (L_X - z)\omega, \quad \omega = \frac{1}{\pi}\tan^{-1}\left(\frac{B}{A}\right). \quad (B\,9)$$

The validation of this solution can be proven by figure B 1. In this diagram we choose two contours $\Gamma_1$ and $\Gamma_2$ both starting from $z^+$ and ending at $z^-$ but $\Gamma_1$ goes anticlockwise while $\Gamma_2$ goes clockwise, where $z^+$ approaches $L$ from the right-hand side of $L$, and $z^-$ approaches $L$ from the left-hand side. Both $z^+$ and $z^-$ approach the same $X_1$ on $L$.

It is obvious that

$$\frac{g(z^+)}{g(z^-)} = \exp(-2i\pi\omega)$$

when we go from $z^+$ to $z^-$ along either $\Gamma_1$ or $\Gamma_2$.

In this case, by taking $\chi(X_1)$ as the unit, equation (B 5) can be rewritten as

$$\Phi^+(X_1) = G(X_1)\Phi^-(X_1) + \frac{\sigma_{12}(X_1)}{|g(X_1)|(A^2 + B^2)^{1/2}}. \quad (B\,10)$$

The solution of equation (B 10) is

$$\Phi(z) = \frac{g(z)}{2\pi i} \int_L \frac{\sigma_{12}(w)dw}{(w - z)g^+(w)(A^2 + B^2)^{1/2}} + P(z)g(z), \quad (B\,11)$$

where $P(z)$ is an arbitrary polynomial and

$$P(z) \propto \Phi(z)z^{-2\omega} \quad \text{when} \quad z \to \infty.$$ 

From equations (B 2) and (B 3) we know that $\Phi(z)$ has the same order as $D(z)$ and $\sigma_{12}(z)$, that is the order of the constant; thus

$$P(z) = \frac{C}{(\Delta a_C - z)^{2\omega}},$$

Figure B 1. The segment $L$ that defines the Hilbert problem.
where \( C \) is a constant, and

\[
g(z)P(z) = \frac{C(L_z - z)^\omega}{(\Delta a_C - z)^\omega}. \tag{B 12}\]

Applying the Plemelj formulae to equation (B 11), we obtain

\[
\Phi^\pm(X_1) = \pm \frac{g^\pm(X_1)\sigma_{12}(X_1)}{2g^\pm(X_1)(A^2 + B^2)^{1/2}} + \frac{g^\pm(X_1)}{2\pi i} \int_{L(w - X_1)g^\pm(w)(A^2 + B^2)^{1/2}} \sigma_{12}(w)\,dw \tag{B 13}
\]

Because \( \Phi^+(X_1) - \Phi^-(X_1) = D(X_1) \), this leads to

\[
D(X_1) = \frac{[g^+(X_1) + g^-(X_1)]\sigma_{12}(X_1)}{2g^+(X_1)(A^2 + B^2)^{1/2}} + \frac{g^+(X_1) - g^-(X_1)}{2\pi i} \int_{L(w - X_1)g^+(w)(A^2 + B^2)^{1/2}} \sigma_{12}(w)\,dw \tag{B 14}
\]

Along \( L \),

\[
g^+(z) = g(X_1) \exp(-2\pi\omega), \quad g^-(z) = g(X_1) \exp(2\pi\omega),
\]

and

\[
g^+(X_1) + g^-(X_1) = g(X_1) \frac{2A}{(A^2 + B^2)^{1/2}}, \quad g^+(X_1) - g^-(X_1) = g(X_1) \frac{2iB}{(A^2 + B^2)^{1/2}}. \tag{B 15}
\]

Substituting equation (B 15) into equation (B 14),

\[
D(X_1) = \beta_1 \sigma_{12}(X_1) + \beta_2 g(X_1) \int_{L} \frac{\sigma_{12}(w)\,dw}{L(w - X_1)g(w)}, \tag{B 16}
\]

where

\[
\beta_1 = \frac{\hat{\alpha}_1}{\bar{\alpha}_3^2 + \pi\alpha_3^2}, \quad \beta_2 = -\frac{\alpha_2}{\bar{\alpha}_3^2 + \pi\alpha_3^2}, \quad \hat{\alpha}_1 = \alpha_1 + \tau_d \text{remote}. \tag{20}
\]

Equation (B 15) is an alternative expression for equation (19) or equations (24a) and (24b).

For the line segment \( L \) defined as \( \{ L : 0 < X_1 \leq \Delta a_C \} \), we take

\[
g(z) = z_\omega^{\omega_1}(\Delta a_C - z)^{\omega_0} \left| (\Delta a_C - z)^{\omega_0 - \omega_1} \right|, \quad \omega_1 = \frac{1}{\pi} \tan^{-1}\left( \frac{B}{A} \right), \tag{B 17}
\]

and repeat the procedure introduced above. We can prove equation (35).

APPENDIX C

When \( \omega_1 + \omega_2 = 1, n \geq 1 \), according to Rosakis (2001) we have the following equations:

\[
\frac{\pi P_n^{(\omega_1, -\omega_2)}(s)}{\sin(\pi\omega_1)} + q^{(\omega_1, \omega_2)}(s) \int_{-1}^{1} \frac{dt}{1 - t} q^{(-\omega_1, -\omega_2)}(t) P_n^{(-\omega_1, -\omega_2)}(t) = \frac{s_0^{(\omega_1, \omega_2)} q^{(\omega_1, \omega_2)}(s) F(n + 1, -n + 1, \omega_1; \frac{1 - x}{2})}{s^{(\omega_1, \omega_2)} q^{(\omega_1, \omega_2)}(s) P_n^{(\omega_1, \omega_2)}(s)}, \tag{59}
\]
where

\[ S_{n, \omega_1, \omega_2}^{(n, \omega_1, \omega_2)} = \frac{\Gamma(-\omega_1)\Gamma(n - \omega_2 + 1)}{\Gamma(n - \omega_1 - \omega_2 + 1)} \left( \frac{n - \omega_2 + 1}{n + 1} \right)^{-1}, \]

when \( \omega_1 > 0, \)

\[ \Gamma(-\omega_1) = \frac{-\pi}{\omega_1 \sin(\pi \omega_1)\Gamma(-\omega_1)}. \]

Because

\[ P_{n}^{(\omega_1, \omega_2)}(x) = \frac{2}{n + \omega_1 + \omega_2} \frac{d}{dx} \left[ P_{n+1}^{(\omega_1-1, \omega_2-1)}(x) \right], \quad (C1) \]

hence

\[ \int_{-x}^{1} P_{n}^{(\omega_1, \omega_2)}(t) \, dt = \frac{2}{n + \omega_1 + \omega_2} \left[ P_{n+1}^{(\omega_1-1, \omega_2-1)}(1) - P_{n+1}^{(\omega_1-1, \omega_2-1)}(x) \right], \quad (58) \]

We also have

\[ q_{n}^{(\omega_1, \omega_2)}(x) = \int_{-x}^{1} \frac{dt}{t} q_{n}^{(-\omega_1, -\omega_2)}(t) P_{n+1}^{(-\omega_1-1, -\omega_2-1)}(t) \]

\[ = q_{n+1, \omega_2+1}(x) \int_{-x}^{1} \frac{dt}{t} q_{n}^{(-\omega_1-1, -\omega_2-1)}(t) P_{n+1}^{(-\omega_1-1, -\omega_2-1)}(t). \quad (C2) \]

This is because, for arbitrary \( \alpha \) and \( \beta, \)

\[ \int_{-1}^{1} \frac{dt}{t} q_{n}^{(-\alpha, \beta)}(t) \frac{(1-t)(1+t)}{(1-t)(1+t)} P_{n}^{(-\alpha-1, \beta-1)}(t) \]

\[ = \int_{-1}^{1} \frac{dt}{-1-t} q_{n}^{(-\alpha-1, \beta-1)}(t) (1-t)(1+t) P_{n}^{(-\alpha-1, \beta-1)}(t) \]

\[ = \int_{-1}^{1} \frac{dt}{-1-t} (1+t) q_{n}^{(-\alpha-1, \beta-1)}(t) P_{n}^{(-\alpha-1, \beta-1)}(t) - \int_{-1}^{1} \frac{dt}{-1-t} (1-t) q_{n}^{(-\alpha-1, \beta-1)}(t) P_{n}^{(-\alpha-1, \beta-1)}(t) \]

\[ = (1-x) \int_{-1}^{1} \frac{dt}{-1-t} (1+t) q_{n}^{(-\alpha-1, \beta-1)}(t) P_{n}^{(-\alpha-1, \beta-1)}(t) + \int_{-1}^{1} (1+t) dt q_{n}^{(-\alpha-1, \beta-1)}(t) P_{n}^{(-\alpha-1, \beta-1)}(t) \]

\[ = (1-x)(1+x) \int_{-1}^{1} \frac{dt}{-1-t} q_{n}^{(-\alpha-1, \beta-1)}(t) P_{n}^{(-\alpha-1, \beta-1)}(t) \]

\[ - (1-x) \int_{-1}^{1} dt q_{n}^{(-\alpha-1, \beta-1)}(t) P_{n}^{(-\alpha-1, \beta-1)}(t) + \int_{-1}^{1} (1+t) dt q_{n}^{(-\alpha-1, \beta-1)}(t) P_{n}^{(-\alpha-1, \beta-1)}(t) \]

\[ = (1-x)(1+x) \int_{-1}^{1} \frac{dt}{-1-t} q_{n}^{(-\alpha-1, \beta-1)}(t) P_{n}^{(-\alpha-1, \beta-1)}(t), \quad (C3) \]

where the following relations have been applied:

\[ \int_{0}^{x} q_{n}^{(\alpha, \beta)}(t) P_{n}^{(\alpha, \beta)}(t) \, dt = P_{n}^{(\alpha+1, \beta+1)}(0) - q_{n}^{(\alpha+1, \beta+1)}(x) P_{n}^{(\alpha+1, \beta+1)}(x) \quad (C4) \]
and
\[
\int_a^b t q^{(\alpha, \beta)}(t) P_n^{(\alpha, \beta)}(t) \, dt = \left[ -t q^{(\alpha + 1, \beta + 1)}(t) P_n^{(\alpha + 1, \beta + 1)}(t) \right]_a^b + \int_a^b q^{(\alpha + 1, \beta + 1)}(t) P_n^{(\alpha + 1, \beta + 1)}(t) \, dt.
\]

(C 5)

By replacing \( P_n^{(-\omega, -\omega)} \) by \( P_n^{(-\omega, -\omega - 1)} \) on the left-hand side of equation (58), and applying equation (C 2), we obtain
\[
- \frac{\pi P_n^{(-\omega, -\omega)}(x)}{\sin (\pi \omega_1)} + q^{(\omega, \omega + 1)}(x) \int_0^1 \frac{dt}{1 - t} q^{(-\omega, -\omega - 1)}(t) P_n^{(-\omega, -\omega - 1)}(t)
\]
\[
= \frac{\pi q^{(\omega, \omega + 1)}(x) F(n + 2, -n + 2, 2 + \omega; \frac{1 - x}{2})}{n + 2}.
\]

(C 6)

where
\[
\mathcal{S}^{(n, \omega_1, \omega_2)} = \left( \frac{n - 1 + \omega_1}{n - 2} \right)^2 \frac{\Gamma(-\omega - 1) \Gamma(n - \omega - 2 + 1)}{2^{\omega_1 + \omega + 1} \Gamma(1 - \omega_1) \Gamma(n + \omega_1 + \omega_2)}, \quad n \geq 2.
\]

(C 7)

According to Bateman (1953),
\[
(1 - x) P_n^{(\omega, \omega)}(x) = \frac{1}{M^{(\omega_1, \omega_2)}} \left[ (n - 1 + \omega_1) P_n^{(\omega_1, 1 + \omega)}(x) - (n - 1) P_n^{(\omega_1 + 1, \omega)}(x) \right],
\]

(C 8)

\[
(1 + x) P_n^{(\omega_1, 1 + \omega)}(x) = \frac{1}{M^{(\omega_1, \omega_2)}} \left[ (n + \omega - 1) P_n^{(\omega_1 + 1, \omega)}(x) + (n - 1) P_n^{(\omega_1 + 1, \omega)}(x) \right],
\]

(C 9)

\[
(1 + x) P_n^{(\omega, \omega + 1)}(x) = \frac{1}{M^{(\omega_1, \omega_2)}} \left[ (n + \omega_2) P_n^{(\omega_1, \omega_2)}(x) + n P_n^{(\omega_1 + 1, \omega)}(x) \right]
\]

(C 10)

and
\[
M^{(\omega_1 + \omega_2)} = n + \frac{\omega_1 + \omega_2}{2};
\]

therefore, we have
\[
\mathcal{S}^{(n, \omega_1, \omega_2)} q^{(1, 1)}(x) P_n^{(\omega_1 + 1, \omega + 1)}(x) = \mathcal{S}^{(n, \omega_1, \omega_2)} p_n^{(\omega_1, \omega_2)}(x) + \lambda_n^{1} P_n^{(\omega_1, \omega + 1)}(x) + \lambda_n^{2} P_n^{(\omega_1 + 1, \omega)}(x),
\]

(C 11)

where
\[
\lambda_n^{0} = -\frac{n(n - 1)}{M^{(\omega_1, \omega_2)}(M^{(\omega_1, \omega_2)} - \frac{1}{2})},
\]

\[
\lambda_n^{1} = \frac{(n - 1 + \omega)(n - 1)}{(M^{(\omega_1, \omega_2)} - \frac{1}{2}) (M^{(\omega_1, \omega_2)} - 1)} - \frac{(n - 1)(n + \omega_2)}{(M^{(\omega_1, \omega_2)} - \frac{1}{2}) (M^{(\omega_1, \omega_2)} - 1)}.
\]

\[
\lambda_n^{2} = \frac{n + \omega - 1(n + \omega_2 - 1)}{(M^{(\omega_1, \omega_2)} - \frac{1}{2}) (M^{(\omega_1, \omega_2)} - 1)}.
\]
Substituting equation (C 11) in equation (C 6), we obtain the following equation:

\[
- \frac{\pi P_{n+1}^{-1, -1}(x)}{\sin(\pi \omega_1)} + \frac{q^{(\omega_1, \omega_2)}(x)}{x-t} \int_{-1}^{1} \frac{dt}{x-t} q^{(-\omega_1, -\omega_2)}(t) P_{n+1}^{(-\omega_1, -\omega_2)}(t) 
\]

\[
= \frac{\pi P_{n+1}^{-1, -1}(x)}{\sin[\pi(\omega_1 + 1)]} + \frac{q^{(\omega_1 + 1, \omega_2 + 1)}(x)}{x-t} \int_{-1}^{1} \frac{dt}{x-t} q^{(-\omega_1 + 1, -\omega_2 + 1)}(t) P_{n+1}^{(-\omega_1 + 1, -\omega_2 + 1)}(t) 
\]

\[
= \sum_{j=0}^{2} \frac{j!}{n!} q^{(\omega_1, \omega_2)}(x) P_{n-j}^{(\omega_1, \omega_2)}(x). 
\]

**APPENDIX D**

Equation (32) includes the definite integrals

\[
I^b_a(x) = \int_a^b q^{(\omega_1, \omega_2)}(w) f(w) \frac{dw}{w-x} = \int_a^b \frac{f(w) \, dw}{a(w-x)(1-w^2)^{1/2}}. 
\]

As listed by Weertman (1996), the solutions for these types of integral are

\[
I_{-1}^1(x) = \int_{-1}^{1} \frac{dw}{(w-x)(1-w^2)^{1/2}} 
\]

\[
= \begin{cases} 
0, & x^2 < 1, \\
\frac{\pi}{4x(1-x^2)^{1/2}}, & x^2 > 1, 
\end{cases} 
\]

\[
I_{-\infty}^{-1}(x) = \int_{-\infty}^{-1} \frac{dw}{(w-x)(1-w^2)^{1/2}} 
\]

\[
= \int_{-\infty}^{-1} \frac{(w+x) \, dw}{(w^2-x^2)(1-w^2)^{1/2}} 
\]

\[
= \begin{cases} 
0, & x^2 < 1, \\
\frac{\pi}{4(1-x^2)^{1/2}} \left(1 + \frac{1}{x}\right), & x^2 > 1. 
\end{cases} 
\]

Thus, within the interval \([-1, 1]\),

\[
\xi_0(x) = -\beta_2 \frac{\text{remote} \, I_{-\infty}^{-1}(x)}{q^{(\omega_1, \omega_2)}(x)} + [u_1(0)] \left( \beta_1 + \frac{\beta_2 I_{-1}^1(x)}{q^{(\omega_1 - 1, \omega_2 - 1)}(x)} \right) 
\]

\[
= \beta_1[u_1(0)]. 
\]

Also

\[
\hat{I}^b_a(x) = \int_a^b \frac{dw}{a(w-x)(1-w^2)^{1/2}} 
\]

\[
= \int_a^b \frac{w \, dw}{a(w^2-x^2)(1-w^2)^{1/2}} + \int_a^b \frac{x \, dw}{a(w^2-x^2)(1-w^2)^{1/2}}, 
\]
where

\[
\int_a^b \frac{w \, dw}{a(w^2 - x^2)(1 - w^2)^{1/2}} = \frac{1}{2(1 - x^2)^{1/2}} \left[ \ln \left( \frac{(1 - x^2)^{1/2} + (1 - w^2)^{1/2}}{(1 - x^2)^{1/2} - (1 - w^2)^{1/2}} \right) \right]_a^b
\]

\[
= \frac{1}{2(1 - x^2)^{1/2}} \left( \ln \left( \frac{(1 - x^2)^{1/2} + (1 - b^2)^{1/2}}{(1 - x^2)^{1/2} - (1 - b^2)^{1/2}} \right) - \ln \left( \frac{(1 - x^2)^{1/2} + (1 - a^2)^{1/2}}{(1 - x^2)^{1/2} - (1 - a^2)^{1/2}} \right) \right),
\]

(D6)

\[
\int_a^b \frac{x \, dw}{a(w^2 - x^2)(1 - w^2)^{1/2}} = \frac{1}{2(1 - x^2)^{1/2}} \left[ \ln \left( \frac{w(1 - x^2)^{1/2} + x(1 - w^2)^{1/2}}{w(1 - x^2)^{1/2} - x(1 - w^2)^{1/2}} \right) \right]_a^b
\]

\[
= \frac{1}{2(1 - x^2)^{1/2}} \left( \ln \left( \frac{b(1 - x^2)^{1/2} + x(1 - b^2)^{1/2}}{b(1 - x^2)^{1/2} - x(1 - b^2)^{1/2}} \right) - \ln \left( \frac{a(1 - x^2)^{1/2} + x(1 - a^2)^{1/2}}{a(1 - x^2)^{1/2} - x(1 - a^2)^{1/2}} \right) \right),
\]

(D7)

Substituting equations (D6) and (D7) into equation (D5),

\[
\bar{F}_{\chi_a/x_c}(x) = \frac{-1}{2(1 - x^2)^{1/2}} \left( \ln \left( \frac{(1 - x^2)^{1/2} + [1 - (x_a/x_c)^2]^{1/2}}{(1 - x^2)^{1/2} - [1 - (x_a/x_c)^2]^{1/2}} \right) + \ln \frac{(x_a/x_c)(1 - x^2)^{1/2} + x[1 - (x_a/x_c)^2]^{1/2}}{(x_a/x_c)(1 - x^2)^{1/2} - x[1 - (x_a/x_c)^2]^{1/2}} \right),
\]

\[
\bar{F}_{-\chi_a/x_c}(x) = \frac{1}{2(1 - x^2)^{1/2}} \left( \ln \left( \frac{(1 - x^2)^{1/2} + [1 - (x_a/x_c)^2]^{1/2}}{(1 - x^2)^{1/2} - [1 - (x_a/x_c)^2]^{1/2}} \right) + \ln \frac{(x_a/x_c)(1 - x^2)^{1/2} + x[1 - (x_a/x_c)^2]^{1/2}}{(x_a/x_c)(1 - x^2)^{1/2} - x[1 - (x_a/x_c)^2]^{1/2}} \right).
\]

References


Cohesive solutions of intersonic moving dislocations